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Three-Dimensional Stresses in a Half Space Caused by Penny-Shaped Inclusions

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Elastic stress fields caused by isotropic penny-shaped inclusions and axisymmetric ellipsoidal inhomogeneties in a semi-infinite solid are investigated. The analytical solution for these problems is obtained by applying Hankel transformations and Eshelby's solution for ellipsoidal inclusions. This new approach can also be applied to other axisymmetric-potential function-related problems in the half space.

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CONTENTS

INTRODUCTION 1

BASIC APPROACH 1

SOLUTION FOR σ'_{ij} 7

ELASTIC STRAIN ENERGY 11

THE ELLIPSOIDAL INHOMOGENEITY 11

SURFACE DISTORTION AND DILATATION FIELD 12

SUMMARY 12

REFERENCES 13

THREE-DIMENSIONAL STRESSES IN A HALF SPACE CAUSED BY PENNY-SHAPED INCLUSIONS

INTRODUCTION

Elastic fields caused by inclusions in infinite media have been extensivley investigated by several authors [1-5] after Eshelby's work [6-8]. Other research efforts have addressed the half-space problem with an inclusion located near the free surface [9-12]. In these studies, the following methods were used: Galerkin vector [9], Papkovich-Neuber displacement potential [10], image stress caused by two cuboidal inclusions with uniform eigenstrains [11], and Green's function in the half space [12]. Mura has recently reviewed these research efforts [13].

When the elastic moduli of an ellipsoidal subdomain of a material differs from those of the remainder (matrix), the subdomain is called an ellipsoidal inhomogeneity. Cracks, voids, and precipitates are examples of these inhomogeneities. A material containing inhomogeneities is assumed to be free from any stress field unless an external stress field σ_{ij}^a is applied. On the other hand, a material containing inclusions is subjected to an internal stress caused by the eigenstrain e_{ij}^T even if it is free from any external loads. The definition of eigenstrains has been given by Mura [13] and is the same as the stress-free-transformation strain described by Eshelby [6].

The solutions for ellipsoidal inhomogeneities can be reduced to the penny-shaped or elliptical crack case by setting the elastic constants λ and μ for the inhomogeneities equal to zero. The solution of the three-dimensional problems for these cracks has received considerable attention [14-19]. The stress field of a penny-shaped crack in the half space can be solved by obtaining the relevant system of integral equations for the problem formulated by Erdogan and Gupta [20] for the stress analysis of multilayered composites with a flaw.

In the present study, Eshelby's method for ellipsoidal inclusions [6-8] and Hankel's transformation method, used to obtain the elastic solutions of a circular dislocation loop in an unbounded media [21] and in the half space [22], are used for the analysis of the elastic solution of axisymmetric inclusions and axisymmetric-ellipsoidal inhomogeneities in the half space. The method provides a novel way for obtaining the image stresses of an ellipsoidal inclusion in the half space. It is used to find a more general solution of an ellipsoidal inclusion with anisotropic eigenstrain. Existing solutions are shown to be special cases of the present result. This method can also be used to obtain the stress field of a penny-shaped crack in the half space.

BASIC APPROACH

In this report, we consider an axisymmetric ellipsoidal inclusion Ω_1 in a half space (Fig. 1). In general, the inclusion Ω_1 is given by

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leq 1. \tag{1}$$

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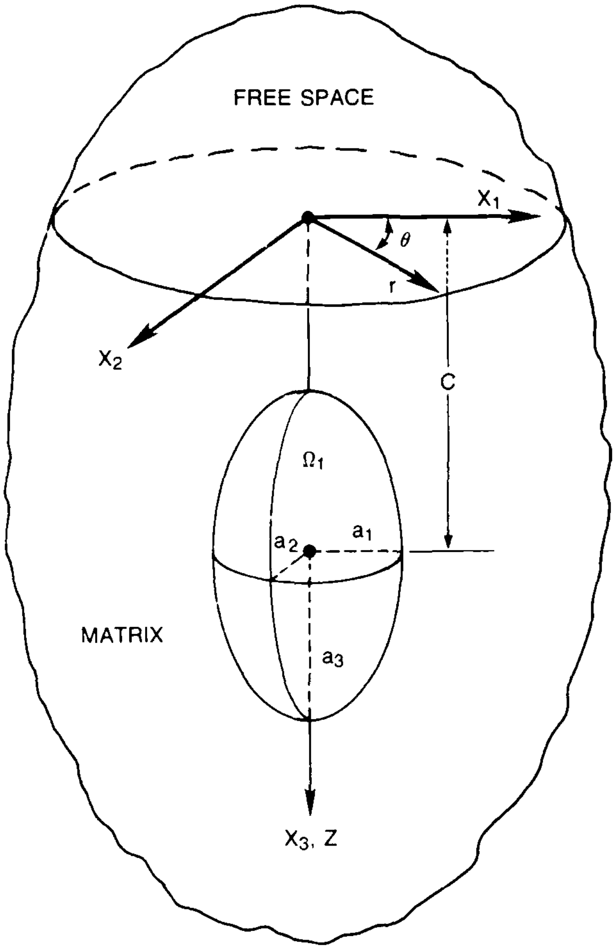

Fig. 1 — Ellipsoidal inclusion with principal half axis $a_1 = a_2$, a_3 in a half space

Symmetry with respect to the x_3-axis is then defined by $a_1 = a_2$, and the anisotropic eigenstrain of the inclusion

$$e_{ij}^T = \delta_{ij}(e + b\delta_{i3}) \qquad i,j = 1, 2, 3, \tag{2}$$

where δ_{ij} is Kronecker delta. (Note that the usual summation convention does not apply to any of the expressions in this report.) Equation (2) states that only normal eigenstrains appear, and $e_{11}^T = e_{22}^T = e$ and $e_{33}^T = e + b$.

For the inclusion Ω_1 defined by Eq. (1) with the uniform eigenstrain described by Eq. (2) with $a_3 \to 0$, the stress field in the unbounded medium outside Ω_1 is obtained by using Eshelby's method [6-8]. The result is given by

$$\sigma_{ij} = \frac{\mu b}{4\pi(1-\nu)} [x_3\phi_{,ij3} - (1-2\nu)(\delta_{i3} + \delta_{j3} - 1)\phi_{,ij} - 2\nu\delta_{ij}\phi_{,33}]$$

$$- \frac{\mu(1+\nu)e}{2\pi(1-\nu)} \phi_{,ij}\,, \tag{3}$$

where the numerical suffixes, $i, j = 1, 2, 3$, following a comma denote differentiation with respect to the Cartesian coordinates x_1, x_2, x_3, e.g. $\phi_{,ij} = \partial^2\phi/\partial x_i \partial x_j$, and φ is the Newtonian potential function that is given by

$$\phi = \pi a_1^2 a_3 \int_{\lambda_0}^{\infty} \frac{U}{\Delta}\, ds , \tag{4}$$

where

$$U = 1 - \left[\frac{x_1^2 + x_2^2}{a_1^2 + s} + \frac{x_3^2}{a_3^2 + s} \right],$$

$$\Delta = (a_1^2 + s)(a_3^2 + s)^{1/2},$$

and λ_0 is the largest root of $U = 0$ outside of Ω_1 and $\lambda_0 = 0$ inside of Ω_1. For inclusions with uniform dilatation eigenstrain only ($b = 0$), Eq. (3) is valid for any a_3 value. The detailed expression of ϕ for both the oblate spheroid ($a_1 > a_3$) and the prolate spheroid ($a_1 < a_3$) are given by Yu [23]. Equation (3) can be transformed into cylindrical coordinates (r, θ, z) as follows:

$$\sigma_{rr} = -\frac{\mu b}{4\pi(1-\nu)} \left[\phi_{,zz} + z\phi_{,zzz} + \frac{1-2\nu}{r}\phi_{,r} + \frac{z}{r}\phi_{,rz} \right] + \frac{\mu(1+\nu)e}{2\pi(1-\nu)} \left(\frac{\phi_{,r}}{r} + \phi_{,zz} \right),$$

$$\sigma_{\theta\theta} = -\frac{\mu b}{4\pi(l-\nu)} \left[2\nu\phi_{,zz} - \frac{1-2\nu}{r}\phi_{,r} - \frac{z}{r}\phi_{,rz} \right] - \frac{\mu(1+\nu)e}{2\pi(1-\nu)} \frac{\phi_{,r}}{r},$$

$$\sigma_{zz} = -\frac{\mu b}{4\pi(1-\nu)} [\phi_{,zz} - z\phi_{,zzz}] - \frac{\mu(1+\nu)e}{2\pi(1-\nu)} \phi_{,zz}, \tag{5}$$

$$\sigma_{rz} = \frac{\mu b}{4\pi(1-\nu)} [z\phi_{,rzz}] - \frac{\mu(1+\nu)e}{2\pi(1-\nu)} \phi_{,rz},$$

$$\sigma_{r\theta} = \sigma_{z\theta} = 0.$$

Equations (5) are obtained with the aid of the following relationships:

$$\nabla^2\phi = 0,$$

$$x_1\phi_{,2} = x_2\phi_{,1},$$

and

$$\phi_{,r} = \frac{1}{r}(x_1\phi_{,1} + x_2\phi_{,2}), \tag{6}$$

where the letter suffixes following a comma denote differentiation with respect to the cylindrical coordinates r, θ, and z, e.g. $\phi,_{rz} = \partial^2\phi/\partial r\partial z$.

For a circular-edge dislocation loop with the z-axis as the axis of symmetry in an unbounded medium (Fig. 2), the stress field is found by Kroupa [21] by using Hankel transformations. For $z > 0$, Kroupa's solution can be rewritten as

$$\sigma_{rr} = -\frac{\mu b'}{2(1-\nu)}\, a\left[(I_0^{-1}),_{zz} + z(I_0^{-1}),_{zzz} + \frac{1-2\nu}{r}(I_0^{-1}),_r + \frac{z}{r}(I_0^{-1}),_{rz}\right],$$

$$\sigma_{\theta\theta} = -\frac{\mu b'}{2(1-\nu)}\, a\left[2\nu(I_0^{-1}),_{zz} - \frac{1-2\nu}{r}(I_0^{-1}),_r - \frac{z}{r}(I_0^{-1}),_{rz}\right],$$

$$\sigma_{zz} = -\frac{\mu b'}{2(1-\nu)}\, a\left[(I_0^{-1}),_{zz} - z(I_0^{-1}),_{zzz}\right], \tag{7}$$

$$\sigma_{rz} = \frac{\mu b'}{2(1-\nu)}\, a\left[z(I_0^{-1}),_{rzz}\right],$$

$$\sigma_{r\theta} = \sigma_{z\theta} = 0,$$

where

$$I_m^n = \int_0^\infty t^n J_m(rt/a)\, J_1(t) e^{-zt/a}\, dt,$$

$$I_m^n = -a(I_m^{n-1}),_z,$$

$$= -ar^{m-1}(r^{-m+1} I_{m-1}^{n-1}),_r \qquad (m = 0, 1, 2\ldots;\ n = -1, 0, 1, 2\ldots),$$

and J_m is the Bessel functin of the mth order, a is the radius of the circular dislocation loop, and b' is the Burger's vector. Equation (7) is obtained by the method of Hankel transformation as used for cylindrically symmetric problems of the theory of elasticity in Sneddon's book [24] and subjected to the following boundary conditions:

$$u_z(r, 0) = \frac{1}{2} b' \text{ for } 0 \le r < a,$$

$$= 0 \text{ for } r > a, \tag{8}$$

$$\sigma_{rz}(r, 0) = 0 \text{ for } 0 \le r < \infty.$$

For the penny-shaped inclusion without shear and dilatation eigenstrains (penny-shaped prismatic inclusion), which is the axisymmetric inclusion when a_3 approaches zero, the equivalent eigenstrains

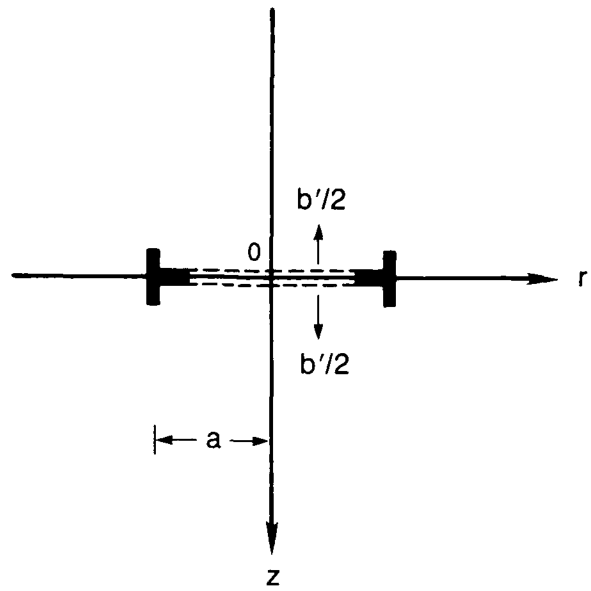

Fig. 2 — Circular edge dislocation loop in infinite solid

are $e_{11}^T = e_{22}^T = 0$, $e_{33}^T \neq 0$. If we reduce Eq. (5) for a penny-shaped prismatic inclusion, that is, $a_3 \to 0$ and $e = 0$, it is interesting to note the similarity between Eqs. (5) and (7). By putting

$$\phi = kI_0^{-1} \qquad (a_3 \to 0), \tag{9}$$

where $k = 2\pi b'a/b$, and $a_1 = a$, the elastic solutions of both the penny-shaped prismatic inclusion (Eq. 5) and the circular-edge dislocation loop (Eq. 7) are identical. This suggests that the method used to investigate the elastic solution of a circular-edge dislocation loop in the half space [22] can be applied to solve the elastic field caused by an axisymmetrical inclusion in the half space. This approach is resonable since the solution of the axisymmetrical inclusion can be obtained by the integration of the results of a penny-shaped prismatic inclusion and the fact that if the inclusion has the same elastic moduli as the matrix, the stress field is the same as that of a small dislocation loop when both the dislocation loop and the inclusion are infinitesimally small [8]. For example, a small inclusion of volume V and an eigenstrain e_{33}^T in the x_3 direction has the same stress field as that of a prismatic interstitial dislocation loop of area A and Burgers vector b_1 provided that $Ve_{33}^T = Ab_1$.

Consider the half space $x_3 = z > 0$ (Fig. 1), an axisymmetric inclusion with the center at the point $(0, 0, c)$ in such a way that its axis of symmetry (z-axis) is perpendicular to the plane of the free surface $z = 0$. In order that the plane $z = 0$ be a free surface, no force must act on it, thus the stress components at $z = 0$ must satisfy the boundary conditions

$$(\sigma_{rz})_{z=0} = 0, \tag{10}$$

$$(\sigma_{zz})_{z=0} = 0,$$

and the equilibrium condition

$$\sum_{j=1}^{3} \sigma_{ij,j} = 0. \tag{11}$$

Similar to the work of Bastecka [22], the stress σ_{ij} outside the axisymmetric ellipsoidal inclusion centered at the point (0, 0, c) but in the half space $z > 0$ is

$$\sigma_{ij} = \sigma_{ij}^{I} + \sigma_{ij}^{II} + \sigma_{ij}' , \qquad (12)$$

which will satisfy the required boundary conditions (Eq. 10) and the equilibrium condition (Eq. 11). This converges to zero for x_1 and x_2 approaching $\pm\infty$ and x_3 approaching ∞. In Eq. (12), the term σ_{ij}^{I} is the stress caused by the axisymmetric inclusion Ω_I (and outside of it) centered at the point (0, 0, c); σ_{ij}^{II} is the stress caused by the image inclusion Ω_2 centered at the point (0, 0, $-c$) (Fig. 3) with eigenstrain

$$(e_{ij}^{T})^{II} = -(e_{ij}^{T})^{I} = -\delta_{ij}(e + b\delta_{i3}) . \qquad (13)$$

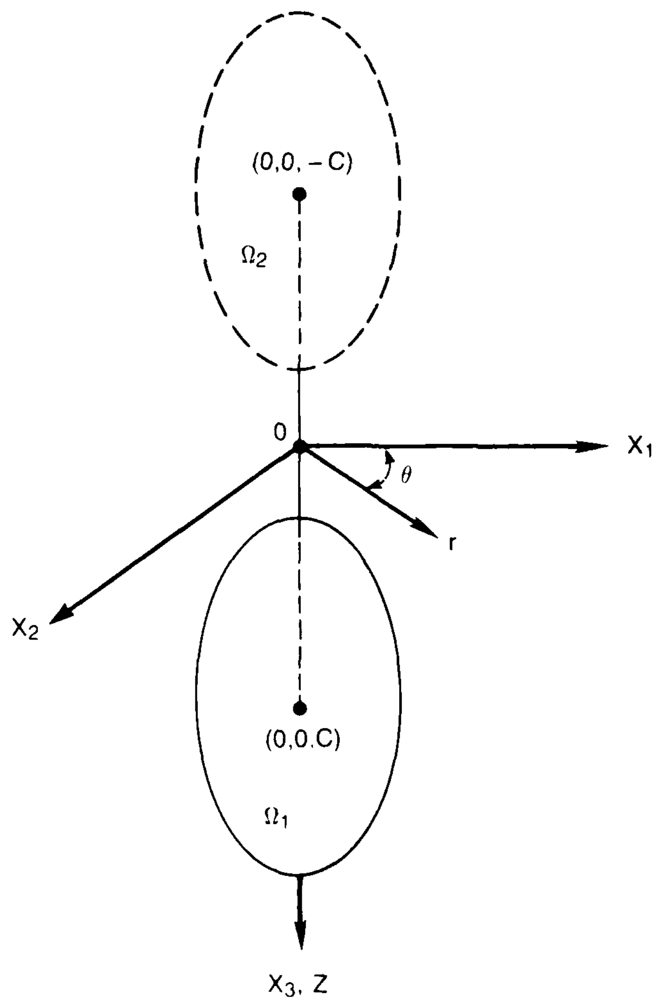

Fig. 3 — Semi-infinite solid containing an ellipsoidal inclusion Ω_1 and its image Ω_2

Equation 13 shows that σ'_{ij} is an additional stress that satisfies the boundary condition

$$(\sigma'_{zz})_{z=0} = -(\sigma^{I}_{zz} + \sigma^{II}_{zz})_{z=0} = 0; \tag{14a}$$

$$(\sigma'_{rz})_{z=0} = -(\sigma^{I}_{rz} + \sigma^{II}_{rz})_{z=0}. \tag{14b}$$

The solutions for the stresses σ^{I}_{ij} and σ^{II}_{ij} are obtained by translating the origin of coordinates in Eq. (3) and Eq. (5) to points (0, 0, c), and (0, 0, $-(c)$ respectively. The Newtonian potential function ϕ^{I} and ϕ^{II} for the solutions of σ^{I}_{ij} and σ^{II}_{ij} respectively are given by

$$\phi^{I} = \pi a_1^2 a_3 \int_{\lambda_1}^{\infty} \frac{U_1}{\Delta}\, ds ,$$

$$\phi^{II} = \pi a_1^2 a_3 \int_{\lambda_2}^{\infty} \frac{U_2}{\Delta}\, ds , \tag{15}$$

where

$$U_1 = 1 - \left[\frac{x_1^2 + x_2^2}{a_1^2 + s} + \frac{(x_3 - c)^2}{a_3^2 + s} \right],$$

$$U_2 = 1 - \left[\frac{x_1^2 + x_2^2}{a_1^2 + s} + \frac{(x_3 + c)^2}{a_3^2 + s} \right],$$

$$\Delta = (a_1^2 + s)\,(a_3^2 + s)^{1/2},$$

and where

λ_1 is the largest root of $U_1 = 0$ for exterior points of Ω_1,

$\lambda_1 = 0$ for interior points of Ω_1, and

λ_2 is the largest root of $U_2 = 0$.

SOLUTION FOR σ'_{ij}

Substituting Eqs. (5) and (15) into Eq. (14) gives

$$(\sigma'_{zz})_{z=0} = 0; \tag{16a}$$

$$(\sigma'_{rz})_{z=0} = \frac{\mu}{2\pi(1-\nu)} \left[cb\phi,^{II}_{rzz} - 2(1+\nu)e\phi,^{II}_{rz} \right]_{z=0}, \tag{16b}$$

where for $z = 0$, $\phi,_{rzz}^{I} = \phi,_{rzz}^{II}$ and $\phi,_{rz}^{I} = -\phi,_{rz}^{II}$. Now, in the limit when a_3 approaches zero, that is, for the penny-shaped inclusion ($a_1 = a_2 = a$), we can substitute Eq. (9) into Eq. (16b) to obtain

$$(\sigma'_{rz})_{z=0} = -\frac{\mu bc}{2\pi(1-\nu)} \frac{k}{a^3} \int_0^\infty t^2 J_1(\rho t) J_1(t) e^{-ct/a} dt$$

$$-\frac{\mu(1+\nu)e}{\pi(1-\nu)} \frac{k}{a^2} \int_0^\infty t J_1(\rho t) J_1(t) e^{-ct/a} dt , \tag{16c}$$

where $\rho = r/a$.

For the axisymmetric problem, by the appropriate expression of the elastic displacements as the derivatives of certair function $\psi(r,z)$ in cylindrical coordinates, the equilibrium and Beltrami equations are replaced by a single equation [24]

$$\nabla^4 \psi(r,z) = 0 , \tag{17}$$

whose general solution is carried out by the method of integral transformations. The function ψ is replaced by its Hankel transform of *zero* th order,

$$G(\zeta,z) = \int_0^\infty r\psi(r,z) J_0(\zeta r)\, dr , \tag{18}$$

and it can be shown that $G(\zeta,z)$ is generally given by the expression

$$G(\zeta,z) = (A + Bz)e^{-\zeta z} + (C + Dz)e^{\zeta z} , \tag{19}$$

where A, B, C and D are unknown functions of ζ, which are determined from the boundary conditions. The stress components are expressed by means of the function $G(\zeta,z)$.

In the present case, we consider the solution to converge to zero for z approaching ∞. Thus we set $C = D = 0$. To determine A and B from the first boundary condition (Eq. 16a), we obtain the following relationship

$$A = -\frac{\mu}{\lambda + \mu} \frac{B}{\zeta} , \tag{20}$$

where $\lambda = 2\mu\nu/(1 - 2\nu)$ is Lame's constant. From the second boundary condition (Eq. 16b), as modified in Eq. (16c), we have

$$\begin{aligned}(\sigma'_{rz})_{z=0} &= f(r) \\ &= \int_0^\infty \zeta F(\zeta) J_1(\zeta r)\, d\zeta ,\end{aligned} \tag{21}$$

where

$$F(\zeta) = -2(\lambda + \mu)\zeta^2 B(\zeta) , \tag{22}$$

and

$$f(r) = -\frac{\mu bc}{2\pi(1-\nu)}\frac{k}{a^3}\int_0^\infty t^2 J_1(\rho t)J_1(t)e^{-ct/a}\,dt$$

$$-\frac{\mu(1+\nu)e}{\pi(1-\nu)}\frac{k}{a^2}\int_0^\infty tJ_1(\rho t)J_1(t)e^{-ct/a}\,dt\,. \tag{23}$$

By letting $t = a\zeta$, Eqs. (21) and (23) give

$$F(\zeta) = -\frac{k\mu}{2\pi(1-\nu)}[cb\zeta + 2(1+\nu)e]J_1(a\zeta)e^{-c\zeta}. \tag{24}$$

By substituting Eqs. (24), (22), and (20) into Eq. (19), the function $G(\zeta,z)$ is found that can then be substituted in the expressions for the stress σ_{ij}' [24, §51]. After substituting the relationship again,

$$\phi^{II} = k(I_0^{-1})^{II} = k\int_0^\infty t^{-1}J_0(\rho t)J_1(t)e^{-t(z+c)/a}\,dt\,, \tag{25}$$

these stresses σ_{ij}' are as follows.

$$\sigma_{rr}' = -\frac{\mu bc}{2\pi(l-\nu)}\left[2\phi,_{zz}^{II} + z\phi,_{zzz}^{II} + \frac{2(1-\nu)}{r}\phi,_{rz}^{II} + \frac{z}{r}\phi,_{rz}^{II}\right]$$

$$+\frac{\mu(1+\nu)e}{\pi(1-\nu)}\left[2\phi,_{zz}^{II} + z\phi,_{zzz}^{II} + \frac{2(1-\nu)}{r}\phi,_{r}^{II} + \frac{z}{r}\phi,_{rz}^{II}\right],$$

$$\sigma_{\theta\theta}' = -\frac{\mu bc}{2\pi(1-\nu)}\left[2\nu\phi,_{zzz}^{II} - \frac{2(1-\nu)}{r}\phi,_{rz}^{II} - \frac{z}{r}\phi,_{rzz}^{II}\right]$$

$$+\frac{\mu(1+\nu)e}{\pi(1-\nu)}\left[2\nu\phi,_{zz}^{II} - \frac{2(1-\nu)}{r}\phi,_{r}^{II} - \frac{z}{r}\phi,_{rz}^{II}\right]. \tag{26}$$

$$\sigma_{zz}' = \frac{\mu bc}{2\pi(1-\nu)}\left[z\phi,_{zzz}^{II}\right] - \frac{\mu(1+\nu)e}{\pi(1-\nu)}\left[z\phi,_{zz}^{II}\right],$$

$$\sigma_{rz}' = \frac{\mu bc}{2\pi(1-\nu)}\left[\phi,_{rzz}^{II} + z\phi,_{rzzz}^{II}\right] - \frac{\mu(1+\nu)e}{\pi(1-\nu)}\left[\phi,_{rz}^{II} + z\phi,_{rzz}^{II}\right],$$

$$\sigma_{r\theta}' = \sigma_{rz}' = 0.$$

When $e = 0$ and $\phi^{II} = k(I_0^{-1})^{II}$, Eq. (26) reduces to the same results obtained by Bastecka [22] for a circular-edge dislocation loop in the half space. In Cartesian coordinates, Eq. (26) becomes

$$\sigma'_{ij} = -\frac{\mu bc}{2\pi(1-\nu)}\Big[(1-2\nu)(\delta_{3i}+\delta_{3j}-1)\phi,^{II}_{ij3} - \phi,^{II}_{ij3} + 2\nu\delta_{ij}\phi,^{II}_{333} - x_3\phi,^{II}_{ij33}\Big] + \frac{\mu(1+\nu)e}{\pi(1-\nu)}\Big[(1-2\nu)(\delta_{3i}+\delta_{3j}-1)\phi,^{II}_{ij} - \phi,^{II}_{ij} + 2\nu\delta_{ij}\phi,^{II}_{33} - x_3\phi,^{II}_{ij3}\Big]. \tag{27}$$

It can be shown that σ'_{ij} satisfies the equation of equilibrium, that is,

$$\sum_{j=1}^{3}\sigma'_{ij,j} = 0. \tag{28}$$

Therefore, for points outside Ω_1, the stress field caused by the presence of a penny-shaped inclusion in the half space can be obtained by Eqs. (3), (12), and (27). Thus,

$$\sigma_{ij} = \frac{\mu b}{4\pi(1-\nu)}\big[(x_3-c)(\phi,^{I}_{ij3} - \phi,^{II}_{ij3}) - (1-2\nu)(\delta_{3i}+\delta_{3j}-1)(\phi,^{I}_{ij} - \phi,^{II}_{ij} + 2c\phi,^{II}_{ij3}) - 2\nu\delta_{ij}(\phi,^{I}_{33} - \phi,^{II}_{33} + 2c\phi,^{II}_{333}) + 2cx_3\phi,^{II}_{ij33}\big] - \frac{\mu(1+\nu)e}{2\pi(1-\nu)}\big[\phi,^{I}_{ij} + \phi,^{II}_{ij} - 2(1-2\nu)(\delta_{3i}+\delta_{3j}-1)\phi,^{II}_{ij} - 4\nu\delta_{ij}\phi,^{II}_{33} + 2x_3\phi,^{II}_{ij3}\big]. \tag{29}$$

For points inside Ω_1, the elastic stress σ^*_{ij} is given by

$$\sigma^*_{ij} = \sigma_{ij} - \sigma^{**}_{ij} = (\sigma^{I}_{ij} - \sigma^{**}_{ij}) + \sigma^{II}_{ij} + \sigma'_{ij} \tag{30}$$

where $-\sigma^{**}_{ij}$ is the uniform stress that exists in the inclusion caused by the uniform eigenstrain e^T_{ij} (Eq. 2). The stress $(\sigma^I_{ij} - \sigma^{**}_{ij})$ is the uniform stress inside the inclusion Ω_1 when the medium is infinite. The solution is expressed explicitly by Mura ([13], Eq. 11.20). Equations (5), (12), and (26) give the stress field in cylindrical coordinates.

Seo and Mura's results [12] for the elastic field in a half space caused by an ellipsoidal inclusion with uniform dilatational eigenstrain (obtained by using Mindlin's solution [25] for Green's function in the half space) can be obtained as a special case by taking $b = 0$ (and $a_1 = a_2$) in Eqs. (29) and (30). Mindlin and Cheng's results [9] for a sphere can also be obtained as a special case by taking $a_1 = a_2 = a_3$ and $b = 0$ in Eq. (29).

ELASTIC STRAIN ENERGY

The elastic strain energy can be expressed as

$$W = -\frac{1}{2}\int_{\Omega_1} \sigma^*_{ij}\, e^T_{ij}\, d\bar{V},$$

$$= -\frac{1}{2}\int_{\Omega_1} \sum_{i=1}^{3} \sigma^*_{ii}\, e d\bar{V} - \frac{1}{2}\int_{\Omega_1} \sigma^*_{33}\, b d\bar{V}, \tag{31}$$

where $\sum_{i=1}^{3} \sigma^*_{ii}$ is the dilation stress field in the inclusion. It is given by

$$\sum_{i=1}^{3} \sigma^*_{ii} = -\frac{2\mu(1+\nu)e}{(1-\nu)}\left[2 - \frac{1}{\pi}(1+\nu)\phi,^{II}_{33}\right]$$

$$-\frac{2\mu(1+\nu)b}{(1-\nu)}\left[1 + \frac{1}{4\pi}(\phi,^{I}_{33} - \phi,^{II}_{33} + 2c\phi,^{II}_{333})\right]. \tag{32}$$

when $b = 0$, the strain energy obtained is the same as that obtained by Seo and Mura [12].

THE ELLIPSOIDAL INHOMOGENEITY

When an inhomogeneity contains an eigenstrain, it is called an inhomogeneity inclusion. Eshelby [6] first pointed out that the stress-field changes caused by an inhomogeneity when the remotely applied stress is σ^a_{ij} can be simulated by the eigenstress caused by an inclusion, if the eigenstrain e^T_{ij} is properly chosen. This eigenstrain is sometimes referred to as the equivalent eigenstrain, or the equivalent stress-free transformation strain. For a given uniformly applied stress σ^a_{ij} and a uniform eigenstrain e^{T*}_{ij}, the normal components of the equivalent eigenstrains e^T_{ij} are given by [23]

$$(\lambda - \lambda^*)e^c + \lambda e^T + 2\mu e^T_{ij} + 2(\mu^* - \mu)\sum_{kl=11}^{33} S_{ijkl}\, e^T_{kl}$$

$$= \lambda^* e^{T*} + (\lambda - \lambda^*)e^a + 2\mu^* e^{T*}_{ij} + 2(\mu - \mu^*)e^a_{ij} \tag{33}$$

where $ij = 11, 22, 33$ and kl denotes summation over 11, 22, 33 only; e^T, e^{T*} and e^a are the sum of three normal components of strains e^T_{ij}, e^{T*}_{ij}, and e^a_{ij} respectively;

$$e^c = \frac{1-2\nu}{4\pi(1-\nu)}(I_1 e^T_{11} + I_2 e^T_{22} + I_3 e^T_{33}) + \frac{\nu}{1-\nu}e^T. \tag{34}$$

In this equation, μ, λ are the elastic constants of the matrix; μ^*, λ^* are the elastic constants of the inhomogeneity; and I_1, I_2, I_3, and S_{ijkl} are constants whose values depend on the shape of the inclusion as given by Eshelby [6-8]. Some detailed expressions for these constants for the inclusions of special shapes are given by Mura [13]. Therefore, by solving the set of three simultaneous equations in Eq. (33), the equivalent eigenstrains e_{11}^T, e_{22}^T, and e_{33}^T are obtained once the uniform eigenstrain e_{ij}^{T*} and uniformly applied stress e_{ij}^a are given. If both e_{ij}^{T*} and e_{ij}^a are axisymmetric for an axisymmetrical inclusion, the resultant equivalent eigenstrain e_{ij}^T is also axisymmetric and can be represented in the form of Eq. (2). Then the results of Eqs. (3), (5), (12), (26), (27), (29), and (30) can be applied accordingly to solve the stress field and strain energy of an axisymmetrical inhomogeneous inclusion in the half space.

SURFACE DISTORTION AND DILATATION FIELD

The roughness of solid surfaces is a second-order effect, but it has profound practical consequences in many fields of engineering and pure science. In many practical situations, the presence of inclusions or inhomogeneities under an external load will change the surface profile. The displacement of the free surface ($z = 0$) solved by the present method is:

$$u_r = \frac{bc}{\pi} (\phi,_{rz}^{II})_{z=0} - \frac{(1+\nu)e}{\pi} (\phi,_r^{II})_{z=0},$$

$$u_z = \frac{b}{2\pi}[(\phi,_z^{II})_{z=0} - c(\phi,_{zz}^{II})_{z=0}] + \frac{e}{\pi} (\phi,_z^{II})_{z=0}. \tag{35}$$

The presence of inclusions or inhomogeneities under an external load will also produce a dilatational field. The dilatational field in the matrix obtained in the present study is:

$$\frac{\Delta V}{V} = -\frac{(1-2\nu)b}{4\pi(1-\nu)} [\phi,_{zz}^{I} - \phi,_{zz}^{II} + 2c\phi,_{zzz}^{II}]$$

$$+ \frac{(1-2\nu)(1+\nu)e}{\pi(1-\nu)} \phi,_{zz}^{II}. \tag{36}$$

The important relationships between the dilatation field and the equilibrium-concentration distribution for dilute solutions in stressed solid are given by Li [26].

SUMMARY

The stress field in the half space ($z \geq 0$) caused by a penny-shaped inclusion Ω_1 centered at $(0, 0, c)$ with eigenstrain $e_{ij}^T = \delta_{ij}(e + b\delta_{i3})$ is found by the superposition of the following three stress fields: (a) the stress field of the inclusion Ω_1 centered at (0,0,c) with eigenstrain e_{ij}^T in an infinite medium; (b) the stress field of the image inclusion Ω_2 centered at $(0, 0, -c)$ with eigenstrain $-e_{ij}^T$; and (c) the additional fictitious stress field that makes all stress fields satisfy the equilibrium and boundary conditions.

The stress field of the penny-shaped prismatic inclusion in an infinite medium obtained by Eshelby is compared with the stress field of a prismatic loop in an infinite medium as obtained by Kroupa [21]. A relationship is found between the potential function ϕ of the inclusion and the integral function I_0^{-1}, which involves the product of the Bessel functions J_m for the solution of the prismatic loop.

The fictitious stress field is solved first for the two-dimensional problem by using the Hankel transformation method and then it is transformed into the three-dimensional case by use of the relationship between ϕ and I_0^{-1}.

The solution of the elastic field in the half space with ellipsodial inclusions with uniform dilatational eigenstrains obtained by Seo and Mura (1979) has been rearranged into three terms corresponding to the sress field of the inclusion Ω_1 in an infinite medium centered at (0,0,-c) with eigenstrain $\delta_{ij}e$, the stress field of the image inclusion Ω_2 centerd at (0,0,-c) with eigenstrain $-\delta_{ij}e$, and the additional fictitious stress field. It has also been shown that when $a_1 = a_2$, Seo and Mura's results are a special case of the present solution.

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